UPPER BOUND ON MOSAIC NUMBER OF KNOTS AND LINKS

HWA JEONG LEE, KYUNGPYO HONG, HO LEE, AND SEUNGSANG OH

Abstract. In [7], Lomonaco-Kauffman introduced a knot mosaic system to give a definition of a quantum knot system which can be viewed as a blueprint for the construction of an actual physical quantum system. A knot n-mosaic is an $n \times n$ matrix of 11 kinds of specific mosaic tiles representing a knot. The mosaic number $m(K)$ of a knot K is the smallest integer n for which K is representable as a knot n-mosaic. In this paper we establish an upper bound on the mosaic number of a knot or a link K in terms of the crossing number $c(K)$. Let K be a nontrivial knot or a non-split link which has at least one nontrivial knot component. Then $m(K) \leq c(K) + 1$. Moreover if K is prime and non-alternating, then $m(K) \leq c(K) - 1$.

1. Knot mosaics

Throughout this paper we will frequently use the term "knot" to mean either a knot or a link for simplicity of exposition. In [7], Lomonaco-Kauffman introduced a knot mosaic system to give a definition of a quantum knot system which can be viewed as a blueprint for the construction of an actual physical quantum system.

Let $\mathbb{T}$ denote the set of the following 11 symbols which is called mosaic tiles;

Let n be a positive integer. We define an *n-mosaic* as an $n \times n$ matrix $M = (M_{i,j})$ of mosaic tiles with rows and columns indexed from 1 to n. A *knot n-mosaic* is an n-mosaic in which each curve segment on a mosaic tile is suitably connected together on both sides with other curve segments on mosaic tiles immediately next to in either the same row or the same column. Then this knot n-mosaic represents a specific knot. One natural question concerning knot mosaics may be to determine the size of matrices representing knots. Define the *mosaic number* $m(K)$ of a knot K as the smallest integer n for which K is representable as a knot n-mosaic. Three examples of knot mosaics in Figure 1 are a 4-mosaic, the Hopf link 4-mosaic, and the trefoil knot 4-mosaic.

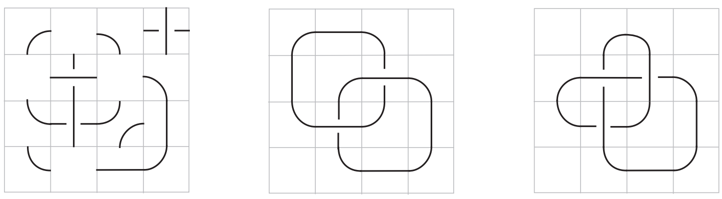

Figure 1. Three examples of 4-mosaics

2010 Mathematics Subject Classification: 57M25, 57M27.

As an analog to the planar isotopy moves and the Reidemeister moves for standard knot diagrams, Lomonaco and Kauffman created for knot mosaics the 11 mosaic planar isotopy moves and the mosaic Reidemeister moves in [7]. They conjectured that for two tame knots (or links) K_1 and K_2, and their arbitrary chosen mosaic representatives M_1 and M_2, respectively, K_1 and K_2 are of the same knot type if and only if M_1 and M_2 are of the same knot mosaic type. This means that tame knot theory and knot mosaic theory are equivalent. Recently Kuriya proved that Lomonaco-Kauffman conjecture is true in [5].

In their paper Lomonaco and Kauffman also proposed several open questions related to knot mosaics. One question is the following; *Is this mosaic number related to the crossing number of a knot?* In this paper we establish an upper bound on the mosaic number of a knot or a link K in terms of its crossing number $c(K)$.

Theorem 1. *Let K be a nontrivial knot or a non-split link which has at least one nontrivial knot component. Then $m(K) \leq c(K) + 1$. Moreover if K is prime and non-alternating, then $m(K) \leq c(K) - 1$.*

Note that the mosaic number of the Hopf link and the connected sum of two Hopf links are 4 and 6, respectively, even though their crossing numbers are 2 and 4, respectively.

2. Arc index and grid diagrams

There is an open-book decomposition of $\mathbb{R}^3$ which has open half-planes as pages and the standard z-axis as the binding axis. We may regard each page as a half-plane H_θ at angle θ when the x-y plane has a polar coordinate. It can be easily shown that every knot K can be embedded in an open-book decomposition with finitely many pages so that it meets each page in a simple arc. Such an embedding is called an *arc presentation* of K. The *arc index* $\alpha(K)$ is defined to be the minimal number of pages among all possible arc presentations of K.

We introduce two theorems which are crucial in the proof of the main theorem. Bae and Park established an upper bound on arc index in terms of crossing number. Corollary 4 and Theorem 9 in [1] provide the following;

Theorem 2. [1] *Let K be a knot or a non-split link. Then $\alpha(K) \leq c(K) + 2$. Moreover if K is prime and non-alternating, then $\alpha(K) \leq c(K) + 1$.*

Later Jin and Park improved the second part of the above theorem as Theorem 3.3 in [4].

Theorem 3. [4] *Let K be a non-alternating prime knot or link. Then $\alpha(K) \leq c(K)$.*

A *grid diagram* is a link diagram of vertical strands and the same number of horizontal strands with the properties that at every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are co-linear and no two vertical segments are co-linear. It is known that every knot admits a grid diagram [2]. The minimal number of vertical segments in all grid diagrams of a knot K is called the *grid index* of K, denoted by $g(K)$. Since grid diagrams are a way for depicting arc presentations [2], we will think of the grid index and the arc index equivalently, i.e. $\alpha(K) = g(K)$. Three figures in Figure 2 show an arc presentation of the trefoil knot, a grid diagram, and how they are related. Note that both of the arc index and the grid index of the trefoil knot are 5.

Dynnikov introduced the following properties of grid diagram system in [3]. In this paper we use cyclic permutations only.

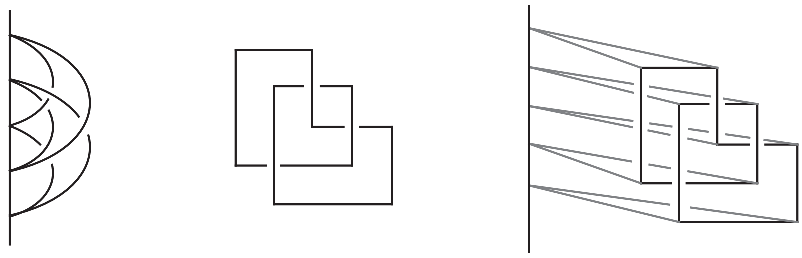

FIGURE 2. An arc presentation and a grid diagram of the trefoil knot

Proposition 4. [3] *Two grid diagrams of the same link can be obtained from each other by a finite sequence of the following elementary moves.*

- *cyclic permutation of horizontal (vertical) edges;*
- *stabilization and destabilization;*
- *interchanging neighbouring edges if their pairs of endpoints do not interleave.*

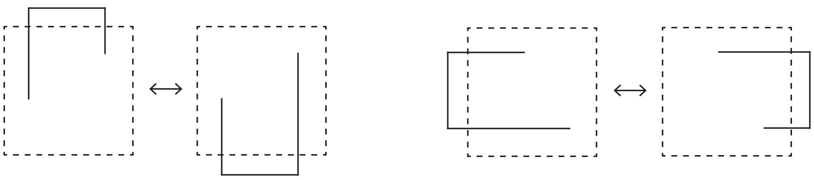

FIGURE 3. Cyclic permutations on a grid diagram

3. Upper bound on the mosaic number

In this section we will prove the main theorem. First we will find an upper bound on the mosaic number in terms of the arc index.

Proposition 5. *Let K be a nontrivial knot or a non-split link which has at least one nontrivial knot component. Then $m(K) \leq \alpha(K) - 1$.*

Proof. Let K be a nontrivial knot or a non-split link which has at least one nontrivial knot component.

We start with a grid diagram with the grid index $g(K)$ (which is equal to $\alpha(K)$). We can regard this grid diagram as a knot mosaic representative of K by smoothing each corner as in Figure 4. This guarantees that $m(K) \leq \alpha(K)$.

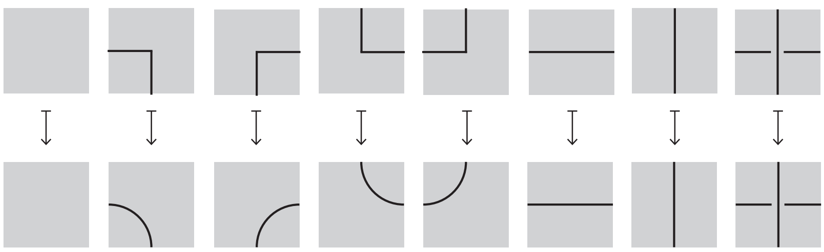

FIGURE 4. From a grid diagram to a knot mosaic representative

Now we will reduce the size by 1. By repeating cyclic permutations of horizontal edges properly, we may assume that the horizontal edge, say h_t, on the top of

this grid diagram is a part of nontrivial knot. Let v_l and v_s be two vertical edges connected to h_t so that v_l is the longer than v_s. If they have the same length, then these must be parts of a trivial knot. Let h_s be the horizontal edge connected to the shorter vertical edge v_s other than h_t, and v_m the vertical edge connected to h_s other than v_s. See Figure 5. By repeating cyclic permutations of vertical edges properly, we can find a grid diagram of K so that v_s lies between v_l and v_m.

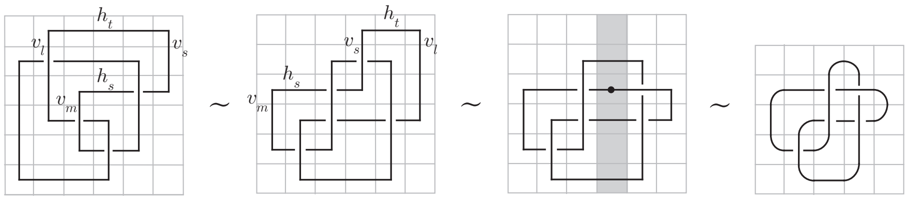

FIGURE 5. Reduction of the size by 1

Next we slide down h_t until it reaches to h_s on the grid diagram, keeping that h_t crosses over vertical edges. Since this sliding uses a combination of planar isotopy moves and Reidemeister moves, it does not change the knot type. Even though it is not a grid diagram anymore, it is still a mosaic representative of K. Finally we can delete one column of mosaic tiles which is the shaded region in the figure. The result is a $(\alpha(K)-1)$-mosaic representative of K. □

Then Theorem 1 follows directly from Theorem 2, 3, and Proposition 5.

4. SHARPER UPPER BOUNDS FOR SEVERAL KNOT CLASSES

In this section we will present sharper upper bounds on the mosaic number of torus knots and pretzel knots.

Corollary 6. *Let $T_{p,q}$ be a (p,q)-torus knot. Then $m(T_{p,q}) \leq p+q-1$. Moreover if $|p-q| \neq 1$, then $m(T_{p,q}) \leq p+q-2$.*

Proof. Figure 6 shows a grid diagram of $T_{p,q}$ with $p < q$ with grid index $p+q$. Now we can follow the proof of Proposition 5 to reduce the size by 1. Moreover if $p+2 \leq q$, then $(p+1)^{th}$ vertical edge and $(p+1)^{th}$ horizontal edge do not share their endpoints. Therefore we can simultaneously apply two sliding moves on the top horizontal edge and on the rightmost vertical edge as in Figure 7. □

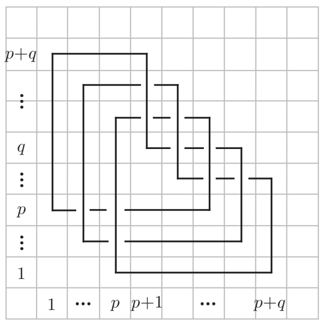

FIGURE 6. Grid diagram of $T_{p,q}$ with grid index $p+q$

Corollary 7. *Let $P(-p,q,r)$ be a pretzel knot of type $(-p,q,r)$ with $p,q,r \geq 2$. Then*

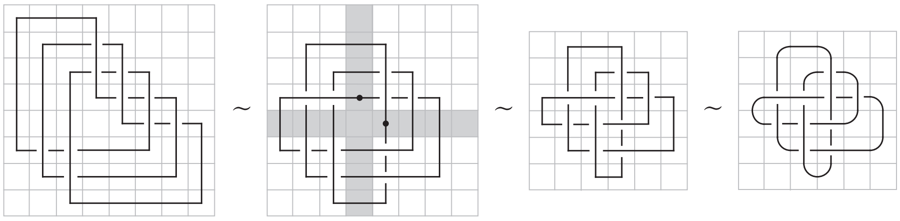

FIGURE 7. Converting a grid diagram of $T_{3,5}$ into a knot 6-mosaic

(1) *If* $K = P(-2, q, r)$ *with* $q, r \geq 3$*, then* $m(K) \leq q + r$*;*
(2) *If* $K = P(-p, 2, r)$ *with* $p, r \geq 3$*, then* $m(K) \leq p + r + 1$*;*
(3) *If* $K = P(-p, 3, r)$ *with* $p, r \geq 3$*, then* $m(K) \leq p + r + 1$*;*
(4) *If* $K = P(-p, q, r)$ *with* $p \geq 3$ *and* $q, r \geq 4$*, then* $m(K) \leq p + q + r - 3$*.*

Proof. This corollary follows directly from the result of [6] combined with Proposition 5. ☐

REFERENCES

[1] Y. Bae and C. Y. Park, *An upper bound of arc index of links*, Math. Proc. Camb. Phil. Soc. **129** (2000) 491–500.

[2] P. R. Cromwell, *Embedding knots and links in an open book I: Basic properties*, Topology Appl. **64** (1995) 37–58.

[3] I. Dynnikov, *Arc-presentations of links: monotonic sumplification*, Fund. Math. **190** (2006) 29–76.

[4] G. T. Jin and W. K. Park, *Prime knots with arc index up to 11 and an upper bound of arc index for nonalternating knots*, J. Knot Theory Ramif. **19** (2010) 1655–1672.

[5] T. Kuriya, *On a Lomonaco-Kauffman conjecture*, Preprint arXiv:0811.0710v3.

[6] H. J. Lee amd G. T. Jin, *Arc index of pretzel knots type of (-p,q,r)*, Preprint arXiv:1204.0597.

[7] S. J. Lomonaco and L. H. Kauffman, *Quantum Knots and Mosaics*, AMS PSAPM **68** (2010), 177–208.

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJEON 305-701, KOREA

E-mail address: hjwith@kaist.ac.kr

DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, ANAM-DONG, SUNGBUK-KU, SEOUL 136-701, KOREA

E-mail address: cguyhbjm@korea.ac.kr

DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, ANAM-DONG, SUNGBUK-KU, SEOUL 136-701, KOREA

E-mail address: stasisfield@korea.ac.kr

DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, ANAM-DONG, SUNGBUK-KU, SEOUL 136-701, KOREA

E-mail address: seungsang@korea.ac.kr